Physics 604 Lecture notes

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Magnetic Shielding Example: A soft iron annular cylinder of magnetic permeability μ has inner radius a and outer radius b and is centered on the z-axis. Far from the cylinder, there is a uniform magnetic field $\mathbf{H} = H_0\hat{x}$. Find the Magnetic Induction $\mathbf{B}$ in all space.

It is convenient to introduce cylindrical coordinates and unit vectors:

$$\begin{aligned}
\rho &= \sqrt{x^2+y^2} \\
\theta &= \tan^{-1}(y/x) \\
\hat{\rho} &= (x\hat{x}+y\hat{y})/\rho \\
\hat{\theta} &= (-y\hat{x}+x\hat{y})/\rho.
\end{aligned}$$

The Gradient and Laplacian operators are:

$$\begin{aligned}
\vec{\nabla}\Phi_M(\rho,\theta,z) &= \left[\hat{\rho}\frac{\partial}{\partial\rho}+\hat{\theta}\frac{1}{\rho}\frac{\partial}{\partial\theta}+\hat{z}\frac{\partial}{\partial z}\right]\Phi_M(\rho,\theta,z) \\
\nabla^2\Phi_M(\rho,\theta,z) &= \left[\frac{1}{\rho}\frac{\partial}{\partial\rho}\rho\frac{\partial}{\partial\rho}+\frac{1}{\rho^2}\frac{\partial^2}{\partial\theta^2}+\frac{\partial^2}{\partial z^2}\right]\Phi_M(\rho,\theta,z)
\end{aligned}$$

In this problem, the free current density vanishes everywhere (except the currents at infinity that generate H_0). It is therefore useful to derive the solution for H and B in terms of a scalar magnetic potential Φ_M:

$$\begin{aligned}
\mathbf{H} &= \nabla\Phi_M \\
\mathbf{B} &= \begin{cases} \mu_0\mathbf{H} & for \quad \rho < a \\ \mu\mathbf{H} & for \quad a < \rho < b \\ \mu_0\mathbf{H} & for \quad b < \rho \end{cases} \\
\nabla^2\Phi_M &= 0, \quad for \quad \rho < a \quad and \quad b < \rho \\
\hat{\rho}\cdot\mathbf{B} & \quad continuous \\
\hat{\theta}\cdot\mathbf{H} & \quad continuous
\end{aligned}$$

Laplace's equation in cylindrical coordinates can be solved by the method of separation of variables. In general (non-zero z-dependence) this will result in the Bessel equation, and Bessel function solutions for the ρ-dependence. In this case there is no z-dependence to either the geometry or the boundary conditions. We will obtain a much simpler solution:

$$\begin{aligned}
\Phi_M(\rho,\theta,z) &= R(\rho)\Theta(\theta)Z(z) \\
Z(z) &= 1 \\
\nabla^2\Phi_M &= \left[\frac{1}{\rho}\frac{\partial}{\partial\rho}\rho\frac{\partial}{\partial\rho}+\frac{1}{\rho^2}\frac{\partial^2}{\partial\theta^2}+\frac{\partial^2}{\partial z^2}\right]R(\rho)\Theta(\theta) \\
&= \left[\frac{1}{\rho}\frac{\partial}{\partial\rho}\rho\frac{\partial}{\partial\rho}+\frac{1}{\rho^2}\frac{\partial^2}{\partial\theta^2}\right]R(\rho)\Theta(\theta) \\
\frac{1}{\Theta(\theta)}\frac{\partial^2}{\partial\theta^2}\Theta(\theta) &= -\frac{1}{R(\rho)}\left[\rho\frac{\partial}{\partial\rho}\rho\frac{\partial}{\partial\rho}\right]R(\rho) = \lambda
\end{aligned}$$

The function $\Theta(\theta)$must be periodic with period 2π. Thus we have solutions and eigenvalues:

$$
\begin{aligned}
\Theta_m(\theta) &= e^{im\theta},\ e^{-im\theta} \quad m = 0, 1, 2, \ldots \\
\lambda_m &= -m^2 \\
\left[\rho\frac{\partial}{\partial\rho}\rho\frac{\partial}{\partial\rho}\right] R_m(\rho) &= -\lambda_m R_m(\rho) \\
R_0(\rho) &= \ln(\rho/\rho_0) \\
R_m(\rho) &= \rho^m, \quad \rho^{-m}, \quad m = 1, 2, \ldots
\end{aligned}
$$

The general (z-independent) solution has the form:

$$
\begin{aligned}
\Phi_M(\rho,\theta) &= \Phi_0 \ln(\rho/\rho_0) \\
&+ \sum_{m=1}^{\infty} \left[\cos(m\theta)\left(a_m\rho^m + b_m\rho^{-m}\right) + \sin(m\theta)\left(c_m\rho^m + d_m\rho^{-m}\right)\right]
\end{aligned}
$$

In our particular case, the boundary condition:

$$
\mathbf{H}(\rho,\theta) \quad \rightarrow \quad H_0\hat{x} = H_0(\hat{\rho}\cos\theta - \hat{\theta}\sin\theta) \;\; as \;\; \rho \rightarrow \infty
$$

truncates the series to only the $m = 1$terms. Furthermore, since $\mathbf{H}$ is finite at infinity, we have both the $\rho^1$and ρ^{-1}terms. On the otherhand, $\mathbf{H}$ must be finite at the origin, so for $\rho < a$ we have only the $\rho\cos(\theta)$term. Our trial solution is:

$$
\begin{aligned}
\Phi_M(\rho,\theta) &= \begin{cases} H_3\rho\cos\theta, & \text{for} \quad \rho < a \\ \left[H_1\rho + H_{-1}\rho^{-1}\right]\cos\theta, & \text{for} \quad a < \rho < b \\ \left[H_0\rho + H_2\rho^{-1}\right]\cos\theta, & \text{for} \quad b < \rho \end{cases} \\
\mathbf{H} &= \nabla\Phi_M \\
&= \begin{cases} H_3\left[\hat{\rho}\cos\theta - \hat{\theta}\sin\theta\right], & \text{for} \quad \rho < a \\ \hat{\rho}\left[H_1 - H_{-1}\rho^{-2}\right]\cos\theta - \hat{\theta}\left[H_1 + H_{-1}\rho^{-2}\right]\sin\theta, & \text{for} \quad a < \rho < b \\ \hat{\rho}\left[H_0 - H_2\rho^{-2}\right]\cos\theta - \hat{\theta}\left[H_0 + H_2\rho^{-2}\right]\sin\theta, & \text{for} \quad b < \rho \end{cases}
\end{aligned}
$$

Notice that I cheated, and put in the boundary condition $\mathbf{H} \rightarrow H_0\hat{x}$ as $\rho \rightarrow \infty$ by hand. Now impose the boundary conditions on $\mathbf{H}$ and $\mathbf{B}$ at $r = a$ and $r = b$ (notice that the continuity of the tangential component of $\mathbf{H}$ is equivalent to the continuity of Φ_M):

$$
\begin{aligned}
\hat{\theta}\cdot\mathbf{H}(\rho = a - \epsilon, \theta) &= \hat{\theta}\cdot\mathbf{H}(\rho = a + \epsilon, \theta) \\
\hat{\theta}\cdot\mathbf{H}(\rho = b - \epsilon, \theta) &= \hat{\theta}\cdot\mathbf{H}(\rho = b + \epsilon, \theta) \\
H_3 &= H_1 + H_{-1}a^{-2} \qquad (1) \\
H_1 + H_{-1}b^{-2} &= H_0 + H_2 b^{-2} \qquad (2) \\
\hat{\rho}\cdot\mathbf{B}(\rho = a - \epsilon, \theta) &= \hat{\rho}\cdot\mathbf{B}(\rho = a + \epsilon, \theta) \\
\hat{\rho}\cdot\mathbf{B}(\rho = b - \epsilon, \theta) &= \hat{\rho}\cdot\mathbf{B}(\rho = b + \epsilon, \theta) \\
\mu_0 H_3 &= \mu\left[H_1 - H_{-1}a^{-2}\right] \qquad (3) \\
\mu\left[H_1 - H_{-1}b^{-2}\right] &= \mu_0\left[H_0 - H_2 b^{-2}\right] \qquad (4)
\end{aligned}
$$

Solve for everything as a function of H_0(e.g.: add equations 2 and 4, equate 1 to [3]/μ_0):

$$
\begin{aligned}
\frac{\mu}{\mu_0}\left[H_1 - H_{-1}b^{-2}\right] &= \left[H_0 - H_2 b^{-2}\right] \\
H_1\left[1 + \frac{\mu}{\mu_0}\right] + H_{-1}\left[1 - \frac{\mu}{\mu_0}\right] b^{-2} &= 2H_0 \\
\frac{\mu}{\mu_0}\left[H_1 - H_{-1}a^{-2}\right] &= \left[H_1 + H_{-1}a^{-2}\right]
\end{aligned}
$$

$$
\begin{aligned}
H_1 \left[\frac{\mu}{\mu_0} - 1\right] &= H_{-1} a^{-2} \left[\frac{\mu}{\mu_0} + 1\right] \\
H_1 \left[1 + \frac{\mu}{\mu_0}\right] - \frac{a^2}{b^2} H_1 \left[\frac{\mu}{\mu_0} - 1\right]^2 \Big/ \left[1 + \frac{\mu}{\mu_0}\right] &= 2H_0 \\
\frac{2H_0 \left[\mu/\mu_0 + 1\right]}{\left[1-(a/b)^2\right]\left[1+(\mu/\mu_0)^2\right] + 2(\mu/\mu_0)\left[1+a^2/b^2\right]} &= H_1 \\
\frac{2H_0 \left[\mu/0 - 1\right] a^2}{\left[1-(a/b)^2\right]\left[1+(\mu/\mu_0)^2\right] + 2(\mu/\mu_0)\left[1+a^2/b^2\right]} &= H_{-1} \\
\frac{4H_0 \left[\mu/\mu_0\right]}{\left[1-(a/b)^2\right]\left[1+(\mu/\mu_0)^2\right] + 2(\mu/\mu_0)\left[1+a^2/b^2\right]} &= H_3 \\
\frac{-H_0 \left[b^2 - a^2\right]\left[(\mu/\mu_0)^2 - 1\right]}{\left[1-(a/b)^2\right]\left[1+(\mu/\mu_0)^2\right] + 2(\mu/\mu_0)\left[1+a^2/b^2\right]} &= H_3
\end{aligned}
$$

The screening factor H_3/H_0 is:

$$
\begin{aligned}
\frac{H_3}{H_0} &= 4\frac{\mu_0/\mu}{\left[1-(a/b)^2\right]\left[1+(\mu_0/\mu)^2\right] + 2(\mu_0/\mu)\left[1+a^2/b^2\right]} \\
&\to \frac{4\mu_0/\mu}{\left[1-(a/b)^2\right]} \quad \text{for} \quad \mu_0/\mu \ll 1
\end{aligned}
$$

Notice that the screening factor is smaller than one only if $\mu_0/\mu < 1/4$. This is because the magnetic shell draws field lines in and thus strengthens the magnetic field for $a < \rho < b$, while at the same time tending to reduce the magnetic field for $\rho < a$.