

## Homework 1, Physics 604 **Solution**

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1.1. A solid sphere of radius  $a$  has total charge  $Q$ . Use Gauss's theorem to obtain the electric field  $\mathbf{E}$  as a function of the distance  $r$  from the center of the sphere, for both  $r < a$  and  $a < r$ .

Consider the following three cases:

- (a) The sphere is a perfect conductor
- (b) The sphere has a uniform charge density
- (c) The sphere has a spherically symmetric charge density  $\rho(r) = \rho_0(r/a)^n$ , with  $n > -3$ .

Use a computer program (excel, etc) to make graphs of the radial component of the electric field  $E_r(r)$  as a function of  $r$ . Make graphs for the first two cases, and for the third case for  $n = -2$  and  $n = +2$ .

(a) Perfect conductor,  $\mathbf{E} = 0$  inside. Outside:

$$\begin{aligned}\int \mathbf{E} \cdot d\mathbf{S} &= 4\pi kQ \\ \int \mathbf{E} \cdot \hat{r} r^2 d\cos\theta d\phi &= 4\pi kQ \\ 4\pi r^2 E_r(r) &= 4\pi kQ \\ E_r(r) &= kQ/r^2\end{aligned}\tag{1}$$

(b) Uniform charge density  $\rho_0 = Q/V$ ,  $V = (4/3)\pi a^3$ . For  $r < a$ :

$$\begin{aligned}\int \mathbf{E} \cdot d\mathbf{S} &= 4\pi kQ(r) \\ 4\pi r^2 E_r(r) &= 4\pi k \int_{r' < r < a} d^3\mathbf{r}' \rho(r') \\ r^2 E_r(r) &= k \int_0^r r'^2 dr' \int_0^\pi \sin\theta d\theta \int_0^{2\pi} d\phi \rho_0 \\ r^2 E_r(r) &= k \frac{4\pi r^3}{\rho_0} \\ E_r(r) &= (4\pi/3)k\rho_0 r \\ &= kQ \frac{r}{a^3}\end{aligned}\tag{2}$$

(c) Spherically symmetric charge distribution:

$$\begin{aligned}Q(r) &= 4\pi \int_0^r r'^2 dr' \rho(r') \\ &= 4\pi \rho_0 \int_0^r r'^2 dr' \left[\frac{r'}{a}\right]^n \rho_0 \\ &= \frac{4\pi \rho_0 a^3}{n+3} \left[\frac{r}{a}\right]^{n+3} \quad \text{for } r \leq a\end{aligned}\tag{3}$$

$$= \frac{4\pi \rho_0 a^3}{n+3} \quad \text{for } r > a.\tag{4}$$

From the denominator, we see the requirement for the restriction  $n > -3$ . From Gauss' law:

$$\begin{aligned}E_r(r) &= kQ(r)/r^2 \\ &= k \frac{4\pi \rho_0 a}{n+3} \left[\frac{r}{a}\right]^{n+1} \quad \text{for } r \leq a\end{aligned}\tag{5}$$

$$= k \frac{4\pi \rho_0 a^3}{n+3} \frac{1}{r^2} \quad \text{for } r > a.\tag{6}$$

1.2. For the three cases of Exercise 1.1, apply the boundary condition on the electrostatic potential  $\phi(r)$ :

$$\phi(r) \rightarrow 0 \quad \text{as } r \rightarrow \infty \quad (7)$$

to obtain a functional expression for  $\phi(r)$  from

$$\phi(r) - 0 = \phi(r) - \phi(\infty) = - \int_{\infty}^r d\mathbf{r} \cdot \mathbf{E} \quad (8)$$

Be careful with the **sign**.

(a) Perfect conductor.  $\phi = \text{constant}$ , inside conductor. For  $r > a$ :

$$\phi(r) = - \int_{\infty}^r \frac{kQ}{r'^2} \hat{r} \cdot \hat{r} dr' \quad (9)$$

$$= +kQ \int_r^{\infty} \frac{dr}{r^2} \quad (10)$$

$$= - \left. \frac{kQ}{r'} \right|_r^{\infty} \quad (11)$$

$$= + \frac{kQ}{r}$$

For  $r \leq a$ ,  $\phi(r) = kQ/a$ .

(b) Uniform charge distribution inside  $a$ . For  $r > a$ , same solution as above. For  $r \leq a$ :

$$\phi(r) = \phi(a) - \int_a^r \mathbf{E}(r') \cdot d\mathbf{r}' \quad (12)$$

$$= \frac{kQ}{a} - \int_a^r E_r(r') dr'$$

$$= \frac{kQ}{a} + \int_r^a kQ \frac{r' dr'}{a^3}$$

$$= \frac{kQ}{a} + \left. \frac{kQ}{2a^3} r'^2 \right|_r^a$$

$$= \frac{kQ}{a} + \frac{kQ}{2a} \left[ 1 - \frac{r^2}{a^2} \right]. \quad (13)$$

Note  $\phi(0) = (3/2)\phi(a)$  is finite.

(c) For  $r > a$ , the solution is the same as before,  $Q = Q(a)$ :

$$\phi(r) = \frac{kQ(a)}{r} = \frac{4\pi k \rho_0 a^3}{n+3} \frac{1}{r^2} \quad (14)$$

For  $r < a$ , integrate again from  $r' = a$  to  $r' = r$ :

$$\begin{aligned} \phi(r) &= \phi(a) - \int_a^r E_r(r') dr' \\ &= \frac{4\pi k \rho_0 a}{n+3} + \frac{4\pi k \rho_0}{n+3} \int_r^a \frac{r'^{n+1}}{a^n} dr' \\ &= \frac{4\pi k \rho_0 a}{n+3} + \left. \frac{4\pi k \rho_0}{n+3} \frac{r'^{n+2}}{(n+2)a^n} \right|_r^a \\ &= k \frac{4\pi \rho_0 a}{n+3} \left[ \frac{n+3}{n+2} - \frac{1}{n+2} \frac{r^{n+1}}{a^{n+1}} \right] \end{aligned} \quad (15)$$

Note that the uniform charge distribution case is obtained in the special case  $n = 0$ .

1.3. A simple capacitor is a device formed by two insulated conductors. If equal and opposite charges  $\pm Q$  are placed on the two conductors, there will be a difference of electrostatic potential  $V$  between the two conductors. The ratio  $|Q/V|$  is the capacitance  $C$ .

- (a) Show that in the Gaussian system of units, the units of capacitance are cm.
- (b) Calculate the capacitance  $C$  of two flat conducting sheets of area  $A$ , separated by a distance  $d \ll \sqrt{A}$ .
- (c) Calculate the capacitance per unit length  $C' = C/L$  of two concentric conducting cylindrical shells. The inner shell has outer radius  $a$  and the outer shell has inner radius  $b > a$ , and  $b - a \ll L$ .
- (d) Explain why it is irrelevant whether or not the inner conductor is a solid cylinder or a hollow cylindrical shell.
- (e) Explain why the **outer** radius of the outer cylinder does not effect the answer

**Solution**

- (a) Use a Gaussian surface in the form of a right cylinder of end-surface area  $a$  with one end inside one the plate carrying positive charge, and the other end in the gap. Only the end-surface in the gap has non-zero electric flux. The electric field points from the plate with + charge to the plate with - charge.

$$\begin{aligned} \int d\mathbf{S} \cdot \mathbf{E} &= 4\pi k\sigma a \\ aE_{\text{gap}} &= 4\pi kQ(a/A) \\ E_{\text{gap}} &= 4\pi kQ/A \end{aligned} \tag{16}$$

The potential difference between the plate at positive charge and the plate at negative charge is:

$$\begin{aligned} \phi(+)-\phi(-) &= -\int_{-}^{+} \mathbf{E} \cdot d\mathbf{x} \\ &= +\int_{+}^{-} (4\pi kQ/A)dx \\ &= +\int_0^d (4\pi kQ/A)dx \\ &= 4\pi kQd/A = Q/C \end{aligned} \tag{17}$$

$$C = \frac{1}{4\pi k} \frac{A}{d} \tag{18}$$

The potential is higher on the surface with positive charge.

In MKS units,  $C = A\epsilon_0/d$ . The unit of capacitance is the Farad (F). The unit of electrostatic potential is the Volt:  $1 \text{ V} = 1 \text{ Joule/Coulomb}$ . One Farad is one Coulomb/Volt =  $C^2/J$ .

In Gauss units,  $k = 1$ ,  $C = A/(4\pi d)$ , Capacitance has unit of length. *n.b., in Gauss units, electrostatic potential has units of statvolt.*

- (b) Answered above
- (c) Coaxial cylindrical capacitor. The inner conductor carries linear charge density  $\lambda = Q/L$ , with  $\lambda$  finite in the limit  $L \rightarrow \infty$ . Use a cylindrical Gaussian surface of length  $l$  and radius  $r$  such that  $a < r < b$ , and coaxial with the capacitor structure. By symmetry, the electric field is radial (in cylindrical coordinates). Therefore the electric flux through the endcaps of our gaussian surface vanishes. We only need to consider the radial surface:

$$\begin{aligned} \int \mathbf{E} \cdot d\mathbf{S} &= 4\pi k\lambda l \\ 2\pi r l E_r(r) &= 4\pi k\lambda l \\ E_r(r) &= \frac{2k\lambda}{r} \end{aligned} \tag{19}$$

Note that unlike the parallel plate case, the electric field is not uniform in between the two co-axial surfaces. The Capacitance  $C$  is defined by

$$\phi(a) - \phi(b) = \frac{Q}{C} \tag{20}$$

Since both  $Q$  and  $C$  go to  $\infty$  in the limit  $L \rightarrow \infty$ , it is better to define the Capacitance per unit length  $C' = C/L$ , which is finite:

$$\phi(a) - \phi(b) = \frac{Q/L}{C/L} = \frac{\lambda}{C'} \quad (21)$$

Integrate  $E_r$  to get  $\phi(r)$ :

$$\begin{aligned} \phi(b) - \phi(a) &= - \int_a^b \mathbf{E} \cdot d\mathbf{r} \\ &= - \int_a^b E_r(r) dr \\ &= -2k\lambda \int_a^b \frac{dr}{r} \end{aligned} \quad (22)$$

$$= -2k\lambda \ln(b/a) = -\lambda/C' \quad (23)$$

$$C' = [2k \ln(b/a)]^{-1} \quad (24)$$

- (d) The derivation of  $E_r$  in the coaxial gap relies upon  $E = 0$  inside the metal. As long as the metal is thicker than the depth  $\delta$  of the surface charge distribution, it does not matter what lies for  $r < a - \delta$  or  $r > b_\delta$ .

1.4. Calculate the force between conductors in the parallel plate capacitor (Problem 1.3b)

- (a) Calculate the force as a function of the distance  $d$  between the plates, if the surface area  $A$  and charge  $Q$  are kept constant.  
 (b) Calculate the force as a function of the distance  $d$  between the plates, if the surface area  $A$  and the potential difference  $V$  are kept constant.

**Solution**

There is a tricky factor of two in this problem. Here are two equivalent techniques.

### Principle of Virtual Work

The force  $F$  on a plate can be calculated by the principle of virtual work. If the plate is displaced a distance  $\delta d$  (positive displacement increases the gap), then the electrostatic force does work

$$W = \mathbf{F} \cdot \delta \mathbf{d} \quad (1)$$

However, the electrostatic force is a conservative force, so the work must be equal to minus the change in potential energy stored in the capacitor:

$$W = -\delta U \quad (2)$$

$$U = \frac{QV}{2} = \frac{Q^2}{2C} = \frac{V^2 C}{2}$$

$$F = -\frac{\delta U}{\delta d} \quad (3)$$

In calculating the change in  $U$  with respect to a change in  $d$ , we must specify whether we are keeping the charge constant (isolated capacitor) or the voltage constant (capacitor connected to battery EMF).

Constant  $Q$ :

$$\begin{aligned} F &= - \left[ \frac{\delta U}{\delta d} \right]_Q = - \left[ \frac{Q^2}{2} \frac{\delta}{\delta d} \frac{1}{C} \right] \\ &= - \left[ \frac{Q^2}{2} \frac{\delta}{\delta d} \left( \frac{d}{A\epsilon_0} \right) \right] \\ &= - \left[ \frac{Q^2}{2A\epsilon_0} \right] = -|Q||E|/2 \end{aligned} \quad (4)$$

$$\mathbf{F}(\text{top}) = Q(\text{top})\mathbf{E}(\text{in gap})/2 \quad (5)$$

The minus sign means the force is attractive (remember, opposite charges attract). We also note that the magnitude of the force is independent of  $d$ , provided  $d^2 \ll A$ .

Constant  $V$ :

$$\begin{aligned}
 F &= - \left[ \frac{\delta U}{\delta d} \right]_V = - \left[ \frac{V^2}{2} \frac{\delta C}{\delta d} \right] \\
 &= - \left[ \frac{V^2}{2} \frac{\delta}{\delta d} \left( \frac{A\epsilon_0}{d} \right) \right] \\
 &= + \left[ \frac{V^2}{2} \frac{A\epsilon_0}{d^2} \right] \\
 &= - \left[ \left( -\frac{V}{d} \right) \frac{1}{2} V \frac{A\epsilon_0}{d} \right] \\
 &= - \left[ E \frac{VC(d)}{2} \right] = -EQ(d)/2
 \end{aligned} \tag{6}$$

The answer is formally the same, but in fact  $Q(d) = VC(d) = VA\epsilon_0/d$ . Thus the force diminishes as  $1/d$ .

### Gauss' Law Method

Consider a small patch of area  $\delta A$  on the plate with charge  $Q$ . By symmetry (for  $d^2 \ll A$ ) the charge  $Q(1 - \delta A/A)$  from the rest of the plate does not exert a force on the charge  $\delta Q = Q\delta A/A$ . Thus we only have to consider the force on  $\delta A$  from the charge on the opposite plate. Consider the electric field generated by the opposite plate alone. This field is symmetric in both directions perpendicular to the plate., but a gaussian surface enclosing a surface  $\delta A$  on the other plate will still enclose a charge  $Q\delta A/A$ . Thus the Electric field in the capacitor gap generated from the charge distribution on just one plate is

$$\begin{aligned}
 E(\text{Single Plate}) &= \frac{\sigma}{2\epsilon_0} \\
 &= \frac{Q}{2C}
 \end{aligned} \tag{7}$$

Thus the force from one plate, acting on a surface element  $\delta A$  on the second plate is

$$\begin{aligned}
 |\delta F| &= \delta Q \frac{Q}{2C} = \frac{Q\delta A}{A} \frac{Q}{2C} \\
 |F| &= \frac{Q^2}{2C}
 \end{aligned} \tag{8}$$

- 1.5. Calculate the force per unit area  $\mathcal{F}$  acting on the inner surface of the outer conductor of the cylindrical capacitor (Problem 1.3c). What is the (local) direction of this force?

Solution

**Principle of Virtual Work** Imagine expanding the inner radius of the outer conductor from  $b$  to  $b + \delta b$ , with constant charge  $Q$  on each conductor. Calculate the electrostatic potential energy using the energy density:

$$\begin{aligned}
 U(a, b) &= \int d^3\mathbf{r} \frac{1}{8\pi k} \mathbf{E}^2(r) \\
 &= \frac{2\pi L}{8\pi k} \int_a^b r dr \left[ \frac{2k\lambda}{r} \right]^2 \\
 &= k\lambda^2 L \int_a^b \frac{dr}{r} = k\lambda^2 L \ln(b/a)
 \end{aligned} \tag{9}$$

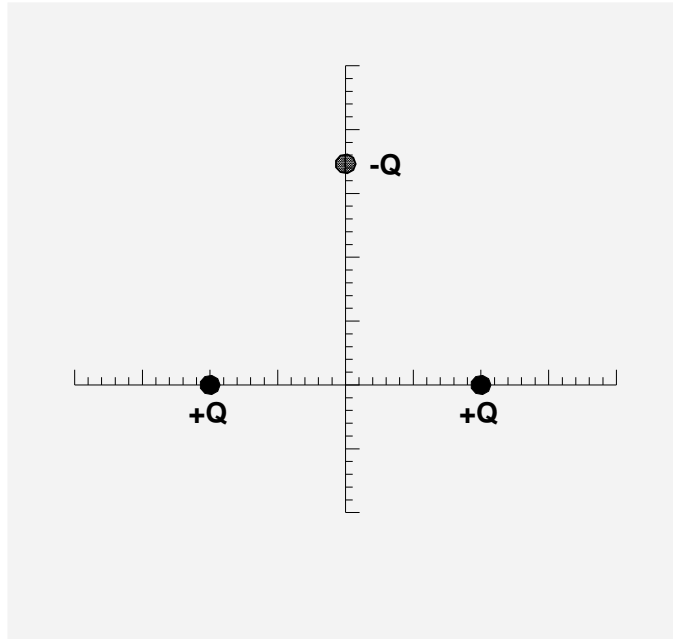
$$U/L = \frac{\lambda^2}{2C'} \tag{10}$$

$$\begin{aligned}
 \mathcal{F}2\pi bL\delta b &= -\frac{\delta U(a, b)}{\delta b} \delta b = -\frac{k\lambda^2 L}{b} \delta b \\
 \mathcal{F} &= -\frac{k\lambda^2}{2\pi b^2}
 \end{aligned} \tag{11}$$

Force density is inward. Total force is zero.

1.6. Three charges in the graph below are arranged as follows:

- $Q_1 = +Q$  at  $(x, y) = (-a, 0)$ .
- $Q_2 = +Q$  at  $(a, 0)$
- $Q_3 = -Q$  at  $(0, a\sqrt{3})$ .



- What is the electrostatic force  $\mathbf{F}_3$  (magnitude and direction, or  $x$ - and  $y$ -components) acting on  $Q_3$  from  $Q_1$  and  $Q_2$ ?
- What is the electric field  $\mathbf{E}(0, 0)$  at the origin (magnitude and direction) from the three charges?
- What is the function dependence of  $\mathbf{E}(0, y)$  as  $y \rightarrow -\infty$ ?
- Is there any point  $(0, y_0)$  such that  $\mathbf{E}(0, y_0) = 0$  (if yes, you do **not** need to solve for  $y_0$ )?
- What is the electrostatic potential energy of this configuration of three charges (not including the energy of each charge in isolation)?

(a) Electrostatic force acting of  $Q_3$ :

$$\mathbf{F}_3 = \mathbf{F}_{3,1} + \mathbf{F}_{3,2} \quad (12)$$

$$= kQ_3Q_1 \frac{\mathbf{r}_3 - \mathbf{r}_1}{|\mathbf{r}_3 - \mathbf{r}_1|^3} + kQ_3Q_2 \frac{\mathbf{r}_3 - \mathbf{r}_2}{|\mathbf{r}_3 - \mathbf{r}_2|^3} \quad (13)$$

$$= -kQ^2 \left[ \frac{a\hat{x} + a\sqrt{3}\hat{y}}{[a^2 + (\sqrt{3}a)^2]^{3/2}} + \frac{-a\hat{x} + a\sqrt{3}\hat{y}}{[a^2 + (\sqrt{3}a)^2]^{3/2}} \right] \quad (14)$$

$$= -kQ^2 \left[ \frac{2a\sqrt{3}\hat{y}}{8a^3} \right] \quad (15)$$

$$= -k \frac{\sqrt{3}Q^2 \hat{y}}{4a^2} \quad (16)$$

(b) Electric Field at origin:

$$\mathbf{E}(0, 0) = \mathbf{E}_1 + \mathbf{E}_2 + \mathbf{E}_3 \quad (17)$$

$$= kQ_1 \frac{-\mathbf{r}_1}{r_1^3} + kQ_2 \frac{-\mathbf{r}_2}{r_2^3} + kQ_3 \frac{-\mathbf{r}_3}{r_3^3} \quad (18)$$

$$= kQ \left[ \frac{a\hat{x}}{a^3} + \frac{-a\hat{x}}{a^3} - \frac{-a\sqrt{3}\hat{y}}{a^3 3^{3/2}} \right] \quad (19)$$

$$= k \frac{Q\hat{y}}{3a^2} \quad (20)$$

- (c) As  $r \rightarrow \infty$ , Electric field approaches point charge limit.  $E(0, y) \rightarrow -kQ\hat{y}/y^2$  as  $y \rightarrow -\infty$ .
- (d) By symmetry, along the  $y$ -axis, the  $x$ -component of  $\mathbf{E}$  vanishes. As  $y \rightarrow \sqrt{3}a$  (from below) the electric field points up towards  $Q_3$ . As  $y \rightarrow -\infty$ , the electric field points down. Somewhere along the  $y$ -axis, the electric field must vanish.
- (e)

$$U = U_{1,2} + U_{1,3} + U_{2,3} \quad (21)$$

$$= kQ_1Q_2 \frac{1}{|\mathbf{r}_1 - \mathbf{r}_2|} + kQ_1Q_3 \frac{1}{|\mathbf{r}_1 - \mathbf{r}_3|} + kQ_2Q_3 \frac{1}{|\mathbf{r}_2 - \mathbf{r}_3|} \quad (22)$$

$$= kQ^2 \left[ \frac{1}{2a} - \frac{1}{2a} - \frac{1}{2a} \right] \quad (23)$$

$$= -kQ^2/(2a) \quad (24)$$