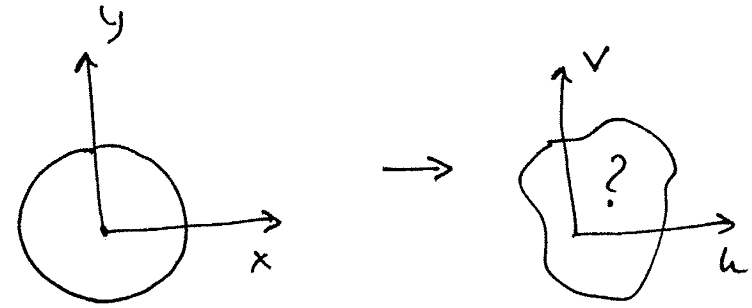
HW 7 solutions

6.7.1.

a) $w(z) = z + \frac{1}{z}$



$x^2 + y^2 = r^2$

$u + iv = x + iy + \frac{x - iy}{x^2 + y^2}$

$\Rightarrow u = x + \frac{x}{x^2+y^2}$, $v = y - \frac{y}{x^2+y^2}$

For a circle $x^2+y^2=r^2 \Rightarrow$ $u = x\left(1 + \frac{1}{r^2}\right)$ $v = y\left(1 - \frac{1}{r^2}\right)$

$\Rightarrow x = \frac{u}{1+r^{-2}}$, $y = \frac{v}{1-r^{-2}}$ and $\left(\frac{u}{1+r^{-2}}\right)^2 + \left(\frac{v}{1-r^{-2}}\right)^2 = r^2$

We can rewrite it as

$\frac{u^2}{a^2} + \frac{v^2}{b^2} = 1$ $a = r + \frac{1}{r}$, $b = r - \frac{1}{r}$ $\Leftrightarrow$ ellipse

b) $w(z) = z - \frac{1}{z}$

$u + iv = x + iy - \frac{x-iy}{x^2+y^2} = x + iy - \frac{x-iy}{r^2}$ for $x^2+y^2=r^2$

$\Rightarrow u = x(1 - r^{-2})$, $v = y(1 + r^{-2})$ $\Rightarrow$ $x = \frac{u}{1-r^{-2}}$, $y = \frac{v}{1+r^{-2}}$

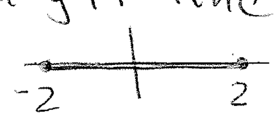
and the eqn $x^2+y^2=r^2$ reduces to

$\left(\frac{u}{r - 1/r}\right)^2 + \left(\frac{v}{r+1/r}\right)^2 = 1$ $\Rightarrow$ ellipse with $a = r - \frac{1}{r}$
$b = r + \frac{1}{r}$

As $|z| \to 1 \Leftrightarrow r \to 1$

a) $r - \frac{1}{r} \to 0$ $\Rightarrow$ $v = \left(r - \frac{1}{r}\right)\sqrt{1 - \frac{u^2}{(r+1/r)^2}} \to 0$

$\Rightarrow$ $v = \emptyset$ $|u| < 2$ - segment of a straight line

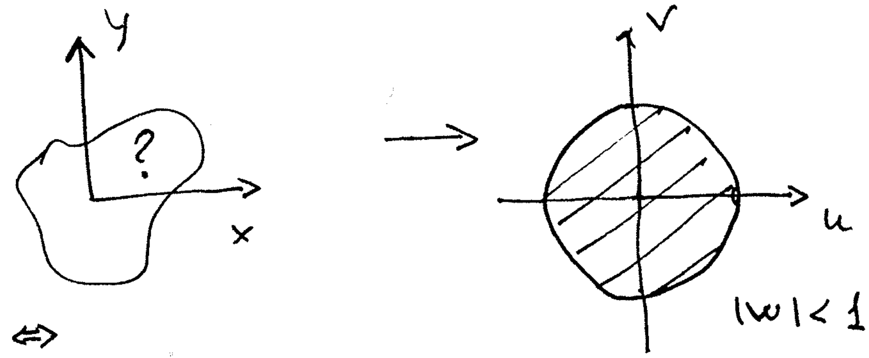


b) $r - \frac{1}{r} \to 0$ $u = \left(r - \frac{1}{r}\right)\sqrt{1 - \frac{v^2}{(r+1/r)^2}} \to 0$

$\Rightarrow$ $u = \emptyset$ $|v| < 2$ - segment of a str. line

6.7.2

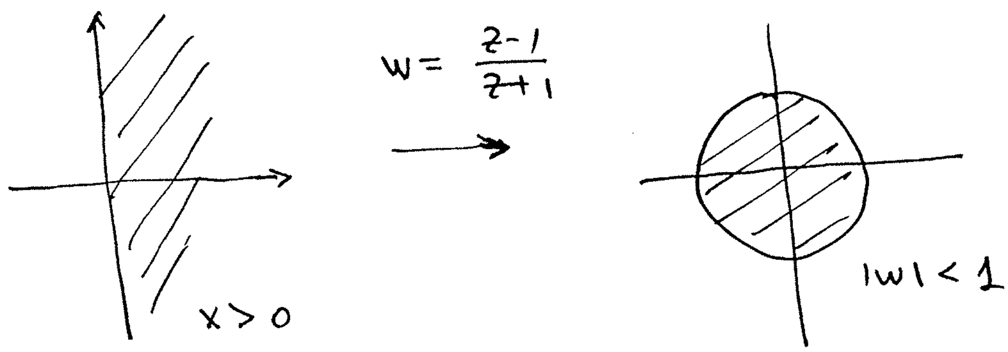
a) $w = \frac{z-1}{z+1}$



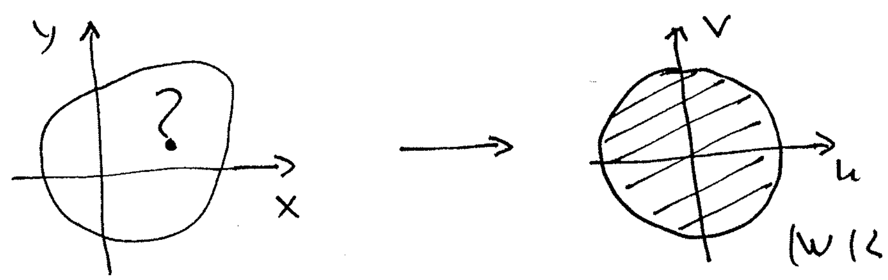
$|w| < 1 \iff ww^* < 1 \iff$

$\iff \frac{z-1}{z+1}\,\frac{z^*-1}{z^*+1} < 1 = \frac{x-1+iy}{x+1+iy}\,\frac{x-1-iy}{x+1-iy} = \frac{(x-1)^2+y^2}{(x+1)^2+y^2} < 1$

$(x-1)^2+\cancel{y^2} < (x+1)^2+\cancel{y^2} \iff \underset{x^2-2x+1}{\overset{\shortparallel}{(x-1)^2}} < \underset{x^2+2x+1}{\overset{\shortparallel}{(x+1)^2}} \iff \underline{x > 0}$



b) $w = \frac{z-i}{z+i}$



$|w| < 1 \iff ww^* < 1$

$ww^* = \frac{z-i}{z+i}\,\frac{z^*+i}{z^*-i} = \frac{x+i(y-1)}{x+i(y+1)}\,\frac{x+i(1-y)}{x-i(y+1)} < 1 \Rightarrow$

$\Rightarrow \frac{x^2+(y-1)^2}{x^2+(y+1)^2} < 1 \iff (y-1)^2 < (y+1)^2 \iff \underline{y > 0}$