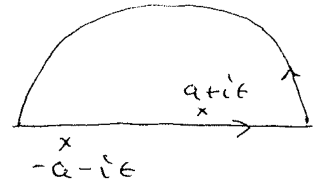
HW 9 solutions

Pr. 1 $\lim_{\epsilon\to 0}\int_{-\infty}^{\infty} dx\, \frac{e^{iax}}{x^2-a^2-i\epsilon} = \left(\begin{matrix}\lim_{\epsilon\to 0}\\ \text{assumed}\end{matrix}\right)\int dx\, \frac{e^{iax}}{x^2-(a+i\epsilon)^2} =$

$= \int dx\, \frac{e^{iax}}{(x-a-i\epsilon)(x+a+i\epsilon)} =$ 

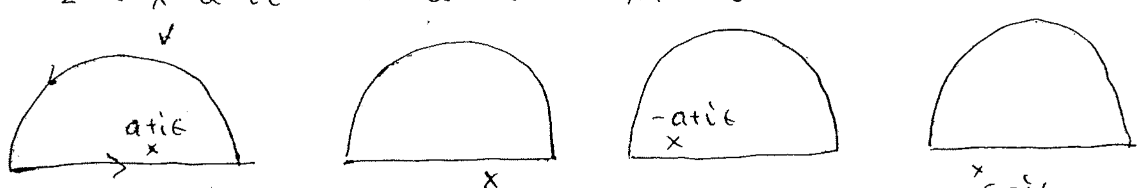
← Jordan's Lemma

$= 2\pi i\, \mathrm{Res}\Big|_{x=a+i\epsilon} \frac{e^{iax}}{(x-a-i\epsilon)(x+a+i\epsilon)} = 2\pi i\, \frac{e^{ia^2}}{2a} = \frac{i\pi}{a} e^{ia^2}$

Pr. 2 $P\int_{-\infty}^{\infty} dx\, \frac{x}{x^2-a^2}\, e^{i\lambda x} \qquad \lambda > 0$

$P\int dx\, \frac{x}{x^2-a^2}\, e^{i\lambda x} = P\int dx\, e^{i\lambda x}\left(\frac{1}{x-a} + \frac{1}{x+a}\right) =$

$= \int dx\, e^{i\lambda x}\, \frac{1}{2}\left(\frac{1}{x-a-i\epsilon} + \frac{1}{x-a+i\epsilon} + \frac{1}{x+a-i\epsilon} + \frac{1}{x+a+i\epsilon}\right)$



$= 2\pi i\, \mathrm{Res}\Big|_{x=a+i\epsilon}$ $\qquad = \emptyset$ $\qquad 2\pi i\, \mathrm{Res}\Big|_{x=-a+i\epsilon}$ $\qquad = \emptyset$

$= \pi i\left(e^{i\lambda a} + \emptyset + e^{-i\lambda a} + \emptyset\right) = \pi i \cos\lambda a$

$(*)\ P\int_{-\infty}^{\infty} dx\, \frac{x}{x^2-a^2} = \lim_{\lambda\to 0} P\int_{-\infty}^{\infty} dx\, \frac{x e^{i\lambda x}}{x^2-a^2} = \pi i$

Warning: the integral $(*)$ is ill-defined

Another calculation:

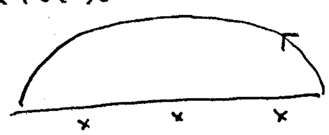
$P\int_{-\infty}^{\infty} dx\, \frac{x}{x^2-a^2} = \lim_{\epsilon\to 0}\left\{ \int_{-\infty}^{-a-\epsilon} dx\, \frac{x}{x^2-a^2} + \underbrace{\cancel{\int_{-a+\epsilon}^{a-\epsilon} dx\, \frac{x}{x^2-a^2}}}_{=0} + \int_{a+\epsilon}^{\infty} dx\, \frac{x}{x^2-a^2} \right\}$ odd function

$= \lim_{\epsilon\to 0}\left\{ -\int_{a+\epsilon}^{\infty} dx\, \frac{x}{x^2-a^2} + \int_{a+\epsilon}^{\infty} dx\, \frac{x}{x^2-a^2} \right\} = \emptyset$

Pr. 3 $P\int_{-\infty}^{\infty} dx \frac{1}{x-a}\frac{1}{x-b}\frac{1}{x-c} = \int_{-\infty} dx\, P\frac{1}{x-a}\, P\frac{1}{x-b}\, P\frac{1}{x-c} =$

$$= \int_{-\infty}^{\infty} dx \left(\frac{1}{x-a+i\epsilon} + i\pi\delta(x-a)\right)\left(\frac{1}{x-b+i\epsilon} + i\pi\delta(x-b)\right)\left(\frac{1}{x-c+i\epsilon} + i\pi\delta(x-c)\right)$$

$$= \int_{-\infty}^{\infty} dx \frac{1}{(x-a+i\epsilon)(x-b+i\epsilon)(x-c+i\epsilon)} + \frac{i\pi}{(a-b)(a-c)} + \frac{i\pi}{(b-a)(b-c)} + \frac{i\pi}{(c-a)(c-b)}$$

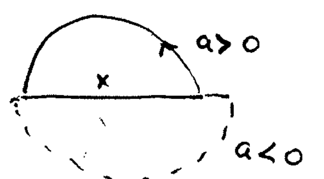

$= \emptyset$

$$= \frac{1}{(a-b)(a-c)(b-c)}\left[i\pi(b-c) - i\pi(a-c) + i\pi(a-b)\right] = \emptyset$$

Pr. 4 $\int_{-\infty}^{\infty} dx \frac{\sin ax}{x} e^{-ibx} = \lim_{\epsilon\to 0}\int_{-\infty} dx \frac{\sin ax}{x-i\epsilon} e^{-ibx} =$

$$= \lim_{\epsilon\to 0}\frac{1}{2i}\int dx \frac{1}{x-i\epsilon}\left(e^{i(a-b)x} - e^{-i(a+b)x}\right)$$

$\int_{-\infty}^{\infty} dx \frac{e^{iax}}{x-i\epsilon} =$



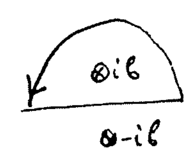
$= \theta(a)\, 2\pi i$

$$\Rightarrow \frac{1}{2i}\int_{-\infty}^{\infty} dx \frac{1}{x-i\epsilon}\left(e^{i(a-b)x} - e^{-i(a+b)x}\right) = \pi\left(\theta(a-b) - \underbrace{\cancel{\theta(-a-b)}}_{=\emptyset}\right)$$

$$= \pi\,\theta(a-b)$$

Pr. 5 $\int_0^{\infty} dx \frac{x\sin ax}{x^2+b^2} = \frac{1}{2}\int_{-\infty}^{\infty} dx \frac{x\sin ax}{x^2+b^2} = \frac{1}{2}\,\mathrm{Im}\int_{-\infty}^{\infty} dx \frac{x e^{iax}}{x^2+b^2}$

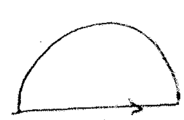
$\int_{-\infty}^{\infty} dx \frac{x e^{iax}}{x^2+b^2} = \int_{-\infty}^{\infty} dx \frac{x e^{iax}}{(x+ib)(x-ib)} =$ Jordan lemma



$$2\pi i\,\mathrm{Res}\big|_{x=ib} \frac{x e^{iax}}{(x+ib)(x-ib)} = 2\pi i \frac{\cancel{i}b e^{-ab}}{2\cancel{i}b} = \pi i e^{-ab}$$

$$\Rightarrow \int_0^{\infty} dx \frac{x\sin ax}{x^2+b^2} = \frac{1}{2}\,\mathrm{Im}\,\pi i e^{-ab} = \frac{\pi}{2}e^{-ab}$$

Pr. 6 $\int_{-\infty}^{\infty} dx \frac{e^{-ikx}}{x+i\epsilon} =$ $k < 0$



$= \emptyset$

$k > 0$

$= -2\pi i\,\mathrm{Res}\big|_{x=\emptyset}$

$$= -2\pi i\,\theta(k)$$