# HW 1 solution

## 5.2.6

a) $\sum_{n=2}^{\infty} \frac{1}{\ln n} \sim \int_2^{\infty} \frac{dx}{\ln x} \Rightarrow$ diverges

b) $\sum \frac{n!}{10^n}$ ratio test: $\lim_{n\to\infty} \frac{a_{n+1}}{a_n} = \lim \frac{n+1}{10} = \infty \Rightarrow$ div

c) $\sum_{n=1}^{\infty} \frac{1}{2n(2n+1)} < \sum_{n=1}^{\infty} \frac{1}{(2n)^2} = \frac{1}{4} \sum_{1}^{\infty} \frac{1}{n^2} \Rightarrow$ conv.

d) $\sum_{n=1}^{\infty} \frac{1}{\sqrt{n(n+1)}} > \sum_{n=1}^{\infty} \frac{1}{n+1} = \sum_{2}^{\infty} \frac{1}{n} \Rightarrow$ diverges

e) $\sum_{n=1}^{\infty} \frac{1}{2n+1} > \sum_{n=1}^{\infty} \frac{1}{2n+2} = \frac{1}{2} \sum_{2}^{\infty} \frac{1}{n} \Rightarrow$ div.

## 5.2.7.

a) $\sum_{1}^{\infty} \frac{1}{n(n+1)} < \sum_{1}^{\infty} \frac{1}{n^2} \sim \int^{\infty} \frac{dx}{x^2} \Rightarrow$ converges

b) $\sum_{n=2}^{\infty} \frac{1}{n \ln n} \sim \int_2^{\infty} \frac{dx}{x \ln x} = \int_{\ln 2}^{\infty} \frac{dy}{y} \Rightarrow$ div.

c) $\sum_{n=1}^{\infty} \frac{1}{n 2^n}$ ratio test $\lim_{n\to\infty} \frac{a_{n+1}}{a_n} = \lim_{n\to\infty} \frac{n}{2(n+1)} = \frac{1}{2} \Rightarrow$

$\Rightarrow$ conv.

d) $\sum_{n=1}^{\infty} \ln\left(1+\frac{1}{n}\right) \sim \int_2^{\infty} dx \ln\left(1+\frac{1}{x}\right) =$

$$= \int_0^{1/2} \frac{dy}{y^2} \ln(1+y) = \underset{\substack{\Downarrow \\ \text{conv.}}}{\int_0^{1/2} \frac{dy}{y^2} [\ln(1+y) - y]} + \underset{\substack{\Downarrow \\ \text{div}}}{\int_0^{1/2} \frac{dy}{y}} \Rightarrow \text{diverges}$$

e) $\sum_{n=1}^{\infty} \frac{1}{n\cdot n^{1/n}}$

Compare $n^{1/n}$ to $\ln n$: $n^{1/n} < \ln n$

Proof: $\underset{\substack{\downarrow \\ \text{decreases}}}{\frac{\ln n}{n}} < \underset{\substack{\downarrow \\ \text{increases}}}{\ln(\ln n)} \Rightarrow \ln n < \ln(\ln n)^n \Rightarrow n < (\ln n)^n$

$\Downarrow$

$n^{1/n} < \ln n$

$\Rightarrow \sum_{n=2}^{\infty} \frac{1}{n\cdot n^{1/n}} > \sum_{n=2}^{\infty} \frac{1}{n \ln n} \sim \int_2^{\infty} \frac{dx}{x \ln x} \Rightarrow$ <u>diverges</u>

$(2n-1)!! = \frac{(2n)!}{2^n n!}$

$(2n)!! = 2^n n!$

5.2.14

$$\sum_1^{\infty} \left[\frac{(2n-1)!!}{(2n)!!}\right]^2 = \sum_1^{\infty} \left[\frac{2n!}{(2n!!)^2}\right]^2 =$$

$$= \sum_1^{\infty} \left[\frac{2n!}{2^{2n}(n!)^2}\right]^2$$

Stirling f-la: $n! \simeq n^{n+1/2} e^{-n} \sqrt{2\pi}$

$2n! \approx (2n)^{2n+1/2} e^{-2n} \sqrt{2\pi}$

$$\simeq \sum_N^{\infty} \left[\frac{2^{2n+1/2}}{2^{2n}} \frac{n^{2n+1/2}}{n^{2n+1}} \frac{e^{-2n}}{e^{-2n}} \frac{\sqrt{2\pi}}{2\pi}\right]^2 = \frac{1}{\pi} \sum_N^{\infty} \frac{1}{n} \Rightarrow \text{diverges}$$

Another solution: Gauss test

$$\frac{a_n}{a_{n+1}} = \left(\frac{2n+2}{2n+1}\right)^2 = \left(\frac{n+1}{n+1/2}\right)^2 = \frac{n^2 + \overset{a_1}{\overset{\shortparallel}{2}}n + 1}{n^2 + \underset{\underset{b_1}{\shortparallel}}{n} + 1/4}$$

$a_1 = b_1 + 1 \Rightarrow$

$\Rightarrow$ <u>div</u>.

5.3.1.

a) $\sum (-1)^n (4n+3) \frac{(2n-1)!!}{(2n+2)!!}$

Leibniz criterion

$$\lim_{n\to\infty} (4n+3) \frac{(2n-1)!!}{(2n+2)!!} = \lim_{n\to\infty} 4n \frac{(2n)!}{2^n n!} \frac{1}{2^{n+1}(n+1)!} =$$

$$= \lim_{n\to\infty} 2^{1-2n} \frac{n(2n)^{2n+1/2} e^{-2n}\sqrt{2\pi}}{(n+1)(n^{n+1/2} e^{-n} \sqrt{2\pi})^2} = \lim_{n\to\infty} \frac{2}{\sqrt{\pi}} \frac{1}{\sqrt{n}} = 0 \Rightarrow \underline{\text{conv.}}$$

b) $\sum (-1)^n (2n+3) \frac{(2n-1)!!}{2n!!}$

Leibniz criterion

$$\lim_{n\to\infty} (2n+3) \frac{(2n-1)!!}{(2n)!!} = \lim_{n\to\infty} 2n \frac{(2n)!}{2^n n!} \frac{1}{2^n n!} =$$

$$= \lim_{n\to\infty} 2^{1-2n} \frac{n(2n)!}{(n!)^2} = \lim_{n\to\infty} 2^{1-2n} \frac{n(2n)^{2n+1/2} e^{-2n}\sqrt{2\pi}}{(n^{n+1/2}e^{-n}\sqrt{2\pi})^2} =$$

$$= \lim_{n\to\infty} 2 \frac{n \cdot n^{2n+1/2}}{n^{2n+1}} = 2\lim_{n\to\infty} \sqrt{n} = \infty \Rightarrow \underline{\text{diverges}}.$$