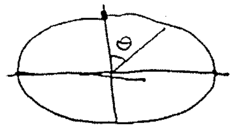
Solutions of HW3

5.8.1.



$$L = \int_0^{\pi/2} d\theta \, \frac{dl}{d\theta}$$

$$dl = \sqrt{dx^2 + dy^2} \Rightarrow \frac{dl}{d\theta} = \sqrt{\left(\frac{dx}{d\theta}\right)^2 + \left(\frac{dy}{d\theta}\right)^2}$$

$$\frac{dx}{d\theta} = a\cos\theta \qquad \frac{dy}{d\theta} = -b\sin\theta \qquad \Rightarrow \frac{dl}{d\theta} = \sqrt{a^2\cos^2\theta + b^2\sin^2\theta} = \sqrt{a^2 - (a^2-b^2)\sin^2\theta}$$

$$= a\sqrt{1 - m\sin^2\theta} \qquad \Rightarrow L = \int_0^{\pi/2} d\theta \, a\sqrt{1-m\sin^2\theta} = a\,E(m).$$

5.8.3

$$K(m) - E(m) = m\int_0^1 dt \, \frac{t^2}{\sqrt{1-t^2}\,\sqrt{1-mt^2}} \Rightarrow$$

$$\Rightarrow \lim_{m\to 0} \frac{K(m)-E(m)}{m} = \int_0^1 dt \, \frac{t^2}{\sqrt{1-t^2}} = \frac{1}{2}\int_0^1 dx \, \frac{\sqrt{x}}{\sqrt{1-x}} = \frac{1}{2}\,\frac{\Gamma(\frac{3}{2})\Gamma(\frac{1}{2})}{\Gamma(2)} = \frac{1}{2}\cdot\frac{1}{2}\cdot\pi = \frac{\pi}{4}$$

5.9.5c

For $f(x) = x^3 \qquad f^{(IV)}(x) = 0 \Rightarrow$

$$\int_0^n f(x)\,dx = \frac{f(0)+f(n)}{2} + \sum_{m=1}^{n-1} f(m) - \frac{1}{2!}\overset{= 1/6}{B_2}\left(f'(n)-f'(0)\right) - \frac{1}{4!}\overset{= -1/30}{B_4}\left(f'''(x)-f'''(0)\right)$$

$$\Rightarrow \sum_{m=1}^{n} f(m) = \int_0^n f(x)\,dx + \frac{f(n)-f(0)}{2} + \frac{1}{2}B_2\left(f'(n)-f'(0)\right) + \frac{1}{4!}B_4\left(f'''(x)-f'''(0)\right)$$

$$f'(x) = 3x^2 \qquad f'''(x) = 6$$

$$\Rightarrow \sum_{m=1}^{n} m^3 = \frac{n^4}{4} + \frac{n^3}{2} + \frac{1}{12}\cdot 3n^2 = \frac{n^2}{4}(n^2+2n+1) = \frac{1}{4}n^2(n+1)^2$$

5.9.10b

$$\ln(1-x) = -\sum_{n=1}^{\infty}\frac{x^n}{n}$$

$$\lim_{a\to 1}\int_0^a \frac{\ln(1-x)}{x}\,dx = -\lim_{a\to 1}\int_0^a dx\,\frac{1}{x}\sum_{n=1}^{\infty}\frac{x^n}{n} = -\lim_{a\to 1}\sum_{n=1}^{\infty}\frac{1}{n}\int_0^a dx\,x^{n-1} =$$

$$= -\lim_{a\to 1}\sum_{n=1}^{\infty}\frac{1}{n^2}(a^n - \cancel{0^n}) = -\lim_{a\to 1}\sum_{n=1}^{\infty}\frac{1}{n^2}a^n = -\sum_{n=1}^{\infty}\frac{1}{n^2} = -\zeta(2)$$

5.9.13

$\lim_{n\to\infty}\zeta(n) = 1 \Rightarrow$ radius of convergence is the same as for series $\sum(-1)^n\frac{1}{n}x^n = \ln(1+x)$ and we know that the series for $\ln(1+x)$ is convergent at $-1 < x \le 1$ (see example 5.6.2)

Proof that $\lim_{n\to\infty}\zeta(n) = 1$:

$$1 + \frac{1}{2^n} + \int_2^{\infty}\frac{dx}{x^n} > \zeta(n) > 1 + \int_2^{\infty}\frac{dx}{x^n} \Rightarrow 1 + \frac{1}{2^n} + \frac{1}{n-1}\,\frac{1}{2^{n-1}} > \zeta(n) > 1 + \frac{1}{(n-1)2^{n-1}} \Rightarrow$$

$$\Rightarrow \lim \zeta(n) = 1$$

5.10.2

$$C(x)=\int_0^x du\cos\frac{\pi u^2}{2} = \underbrace{\int_0^\infty du \cos\frac{\pi u^2}{2}}_{=\frac12} - \int_x^\infty du\cos\frac{\pi u^2}{2} = \frac12 - \frac{1}{\sqrt{2\pi}}\int_{\frac{\pi x^2}{2}}^\infty \frac{dt}{\sqrt t}\cos t$$

$$\phi(z)=\int_z^\infty \frac{dt}{\sqrt t}\cos t$$

$$\phi(z) = \text{by parts} = \int_z^\infty dt\, \frac{1}{\sqrt t}\frac{d}{dt}\sin t = \frac{\sin t}{\sqrt t}\Big|_z^\infty + \frac12\int_z^\infty dt\, \sin t\, t^{-3/2} =$$

$$= -\frac{\sin z}{\sqrt z} + \frac12\int_z^\infty dt\, t^{-3/2}\left(-\frac{d}{dt}\cos t\right) = -\frac{\sin z}{\sqrt z} - \frac12 \frac{\cos z}{z^{3/2}}\Big|_z^\infty + \frac12\int_z^\infty dt\cos t\, \frac{d}{dt} t^{-3/2}$$

$$= -\frac{\sin z}{\sqrt z} + \frac12\frac{\cos z}{z^{3/2}} - \frac12\cdot\frac32\int_z^\infty dt\, t^{-5/2}\underbrace{\cos t}_{=\frac{d}{dt}\sin t} = -\frac{\sin z}{\sqrt z}+\frac12\frac{\cos z}{z^{3/2}} - \frac12\cdot\frac32\cdot\frac{\sin t}{t^{5/2}}\Big|_z^\infty$$

$$+\frac12\cdot\frac32\int_z^\infty dt\, \sin t\, \frac{\partial}{\partial t}t^{-5/2} = -\frac{\sin z}{\sqrt z}+\frac12\cos z\, z^{-3/2} + \frac12\cdot\frac32\,\frac{\sin z}{z^{5/2}} - \frac12\cdot\frac32\cdot\frac52\int_z^\infty dt$$

$$\underbrace{\sin t}_{=-\frac{d}{dt}\cos t}\, t^{-7/2} = -\frac{\sin z}{\sqrt z}+\frac12\frac{\cos z}{z^{3/2}} + \frac12\cdot\frac32\,\frac{\sin t}{z^{5/2}} + \frac12\cdot\frac32\cdot\frac52\,\frac{\cos t}{t^{7/2}}\Big|_z^\infty + \frac12\cdot\frac32\cdot\frac52\cdot\frac72\cdot\int_z^\infty dt\, t^{-9/2}\underbrace{\cos t}_{=-\frac{d}{dt}\sin t} =$$

$$= -\frac{\sin z}{\sqrt z} + \frac12\frac{\cos z}{z^{3/2}} + \frac12\cdot\frac32\,\frac{\sin z}{z^{5/2}} - \frac12\cdot\frac32\cdot\frac52\,\frac{\cos z}{z^{7/2}} + \dots$$

$$+\frac12\cdot\frac32\cdot\frac52\cdot\frac72\,\frac{\sin z}{z^{9/2}} - \frac12\cdot\frac32\cdot\frac52\cdot\frac72\cdot\frac92\int_z^\infty dt\, t^{-11/2}\sin t$$

$$= -\frac{\sin z}{\sqrt z}\left(1 - \frac12\cdot\frac32\,\frac{1}{z^2} + \frac12\cdot\frac32\cdot\frac52\cdot\frac72\,\frac{1}{z^4}+\dots\right) + \frac{\cos z}{z^{3/2}}\left(\frac12 - \frac12\cdot\frac32\cdot\frac52\cdot\frac{1}{z^2}+\right.$$

$$\left.+\frac12\cdot\frac32\cdot\frac52\cdot\frac72\cdot\frac92\,\frac{1}{z^4}+\dots\right) = -\frac{\sin z}{\sqrt z}\sum_{k=0}^\infty(-1)^k\frac{(4k-1)!!}{(2z)^{2k}} + \frac{\cos z}{2z^{3/2}}\sum_{k=0}^\infty (-1)^k\frac{(4k+1)!!}{(2z)^{2k}}$$

$$\Rightarrow \phi(z)\approx -\frac{\sin z}{\sqrt z}\sum_{k=0}^\infty(-1)^k\frac{(4k-1)!!}{(2z)^{2k}} + \frac{\cos z}{\sqrt z}\sum_{k=0}^\infty(-1)^k\frac{(4k+1)!!}{(2z)^{2k+1}}$$

$z=\frac{\pi x^2}{2}$

$$C(x)\approx\frac12 - \frac{1}{\sqrt{2\pi}}\phi\left(\frac{\pi x^2}{2}\right) = \frac12 + \frac{1}{\pi x}\sin\frac{\pi x^2}{2}\sum_{k=0}^\infty(-1)^k\frac{(4k-1)!!}{(\pi x^2)^{2k}} - \frac{1}{\pi x}\cos\frac{\pi x^2}{2}\sum_{k=0}^\infty\frac{(-1)^k(4k+1)!!}{(\pi x^2)^{2k+1}}$$

Similarly,

$$S(x)\approx\frac12 + \frac{1}{\pi x}\cos\frac{\pi x^2}{2}\sum_{k=0}^\infty(-1)^k\frac{(4k-1)!!}{(\pi x^2)^{2k}} - \frac{1}{\pi x}\sin\frac{\pi x^2}{2}\sum_{k=0}^\infty(-1)^k\frac{(4k+1)!!}{(\pi x^2)^{2k+1}}$$