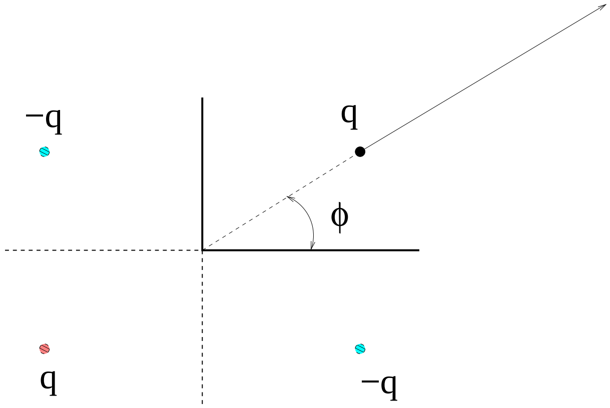
604 Midterm (20 points). 10/16/07, 5:45 ÷ 7:15 p.m.

Problem 1 .

Two semi-infinite grounded conducting plates meet at right angles. How much work does it take to bring a point charge q from infinity to the point located at distance a from the first plate and distance b from the second?

Solution # 1

It is easy to see that the two conducting planes can be replaced by three image charges as shown below



The force acting on the charge is

$$\vec{F} = \frac{q^2}{4\pi\epsilon_0}\Big[\frac{1}{4a^2}\hat{e}_1 + \frac{1}{4a^2}\hat{e}_2 - \frac{1}{4c^2}(\hat{e}_1\cos\varphi + \hat{e}_2\sin\varphi)\Big]$$

where I've introduced notations $c = \sqrt{a^2+b^2}$ and $\varphi = \arctan\frac{y}{x}$. Let us bring the charge from infinity along the path shown above. The image charges will move with the original chagre according to symmetry so $\varphi = const$ along the path. The corresponding work is

$$W = \int_\infty^{a,b}\vec{F}\cdot\vec{dl} = -\frac{q^2}{4\pi\epsilon_0}\int_c^\infty dr\Big[\frac{1}{4r^2\cos\varphi} + \frac{1}{4r^2\sin\varphi} - \frac{1}{4r^2}\Big] = -\frac{q^2}{16\pi\epsilon_0}\Big[\frac{1}{a} + \frac{1}{b} - \frac{1}{c}\Big]$$

Solution # 2

Use formula

$$W = \frac{1}{2}\sum_i q_i\phi(\vec{r}_i)$$

In our case, the induced charges do not contribute since the potential at the surface of the conductor is 0 so we get only the contribution from the point charge

$$W = \frac{1}{2}q\phi(\vec{r})$$

where $\phi(\vec{r})$ is the potential due to all other charges (= the potential due to image charges)

$$W = \frac{q}{2}\phi(\vec{r}) = \frac{q}{8\pi\epsilon_0}\Big[-\frac{q}{2a} - \frac{q}{2b} + \frac{q}{2c}\Big] = -\frac{q^2}{16\pi\epsilon_0}\Big[\frac{1}{a} + \frac{1}{b} - \frac{1}{c}\Big]$$

Solution # 3

By symmetry, the work required to bring the charge to the conducting plane is equal to 1/4 of the work required to assemble all charges (real and image). The latter has the form

$$W_{\text{all charges}} = \frac{q^2}{4\pi\epsilon_0}\Big[-2\frac{q^2}{2a} - 2\frac{q^2}{2b} + 2\frac{q^2}{2c}\Big]$$

and therefore

$$W = \frac{1}{4}W_{\text{all charges}} = \frac{q^2}{16\pi\epsilon_0}\Big[-\frac{q^2}{a} - \frac{q^2}{b} + \frac{q^2}{c}\Big]$$

Problem 2.

The potential of the $y > 0$ half-plane of the XY plane is maintained at $\phi = V$ while that of $y < 0$ half at $\phi = -V$. Find the potential at $z > 0$ (assume that there are no charges at $z > 0$).

Solution # 1: separation of variables.

Separating the variables s and φ, we obtain the solution of the Laplace eqn in a two-dimensional corner in the form

$$(a_\nu s^\nu + b_\nu s^{-\nu})(A_\nu \cos\nu\varphi + B_\nu \sin\nu\varphi), \qquad \nu \neq 0$$

$$(a_0 + b_0 \ln s)(A_0 + B_0\varphi) \qquad \nu = 0$$

Consider at first $\nu \neq 0$. From the boundary condition at $\varphi = 0$ we see that $(a_\nu s^\nu + b_\nu s^{-\nu})A_\nu = V$ which cannot be satisied unless $\nu = 0$. At $\nu = 0$ the boundary conditions are

$$(a_0 + b_0 \ln s)A_0 = V, \quad (a_0 + b_0 \ln s)(A_0 + B_0\pi) = -V$$

which gives $b_0 = 0$, $a_0A_0 = V$, $a_0B_0 = -\frac{2V}{\pi}$ so the solution is

$$\phi \ = \ V\Big[1 - \frac{2}{\pi}\varphi\Big]$$

Is is easy to see that the above expression satisfies Laplace eqn with our boundary conditions.

Solution # 2: Green's function.

The general solution of the Dirichlet problem is

$$\phi(\vec{r}) = \frac{1}{4\pi\epsilon_0}\int d^3r' G_D(\vec{r},\vec{r'})\rho(\vec{r'}) - \frac{1}{4\pi}\int da\phi(\vec{r'})\frac{\partial G_D(\vec{r},\vec{r'})}{\partial n'}$$

In our case, $\rho = 0$ above the $z = 0$ plane and $\vec{n}' = -\hat{e}_3$ so we get

$$\phi(\vec{r} = \frac{1}{4\pi}\int dx'dy' \ \phi(\vec{r'})\frac{\partial G_D(\vec{r},\vec{r'})}{\partial z'}$$

The Dirichlet Green function for the upper half-plane is

$$G_D(\vec{r},\vec{r'}) = 1\sqrt{(x-x')^2+(y-y')^2+(z-z')^2} - 1\sqrt{(x-x')^2+(y-y')^2+(z-z')^2}$$

and therefore

$$\frac{\partial}{\partial z'}G_D(\vec{r},\vec{r'}) = \frac{2z}{[(x-x')^2+(y-y')^2+z^2]^{3/2}}$$

so

$$\phi(\vec{r}) = \frac{V_0 z}{2\pi}\int_0^\infty dx'\int_{-\infty}^\infty dy'\frac{1}{[(x-x')^2+(y-y')^2+z^2]^{3/2}} - \frac{V_0 z}{2\pi}\int_{-\infty}^0 dx'\int_{-\infty}^\infty dy'\frac{1}{[(x-x')^2+(y-y')^2+z^2]^{3/2}}$$

Making shift $y' - y \to y'$ and taking integral over y' we get

$$\phi(\vec{r}) = \frac{V_0 z}{\pi}\int_0^\infty dx'\frac{1}{(x-x')^2+z^2} - \frac{V_0 z}{2\pi}\int_{-\infty}^0 dx'\frac{1}{(x-x')^+z^2}$$
$$= \frac{V_0}{\pi}\int_{\frac{x}{z}}^\infty \frac{dt}{t^2+1} - \frac{V_0}{\pi}\int_{-\infty}^{\frac{x}{z}}\frac{dt}{t^2+1} = \frac{V_0}{\pi}\Big[\frac{\pi}{2} - \arctan\frac{x}{z}\Big] - \frac{V_0}{\pi}\Big[\frac{\pi}{2} + \arctan\frac{x}{z}\Big] = \frac{V_0}{\pi}(\pi - 2\phi),$$

same as above ($\varphi = \arctan\frac{z}{x} = \frac{\pi}{2} - \arctan\frac{x}{z}$).

Problem 3.

The potential at the surface of the sphere (radius R) is given by

$$V_0(\theta) \ = \ k\cos 3\theta$$

Find the potential inside and outside the sphere. (Assume that there is no charge inside or outside the sphere)

Solution

The general solution with azimuthal symmetry has the form

$$\phi(r,\theta) = \sum_{l=0}^{\infty}(A_l r^l + B_l r^{-l-1})P_l(\cos\theta)$$

We need the expansion of $\cos 3\theta$ in Legendre polynomials:

$$\cos 3\theta = \Re(\cos\theta + i\sin\theta)^3 = \cos^3\theta - 3\cos\theta\sin^2\theta = \frac{8}{5}P_3(\cos\theta) - \frac{3}{5}\cos\theta$$

The potential inside the sphere must be regular as $r \to 0$ so

$$\phi_{\text{in}}(r,\theta) = \sum_{l=0}^{\infty} A_l r^l P_l(\cos\theta)$$

From the boundary condition

$$\phi(r=R) = \frac{8k}{5}P_3(\cos\theta) - \frac{3k}{5}\cos\theta$$

we see that $A_3 = \frac{8k}{5R^3}$ and $A_1 = -\frac{3k}{5R}$ while all other A_l vanish. Thus, the solution inside the sphere has the form

$$\phi_{\text{in}}(r,\theta) = \frac{8k}{5}\frac{r^3}{R^3}P_l(\cos\theta) - \frac{3k}{5}\frac{r}{R}\cos\theta$$

Similarly, for the solution outside of the sphere we get

$$\phi_{\text{in}}(r,\theta) = \frac{8k}{5}\frac{R^4}{r^4}P_l(\cos\theta) - \frac{3k}{5}\frac{R^2}{r^2}\cos\theta$$