Final Exam – Spring 2001

Due Tuesday 1 May at 5 p.m. in my mailbox

Problem 1 (8 points)

Prove that the anticommutator of the free Dirac fields $\{\hat{\psi}(x), \hat{\bar{\psi}}(y)\}$ vanishes at space-like separations. (Here $x = (\vec{x}, t)$ and $y = (\vec{y}, t')$ are two arbitrary 4-vectors).

Problem 2 (12 points)

The theory of charged π-mesons interacting with photons is determined by the Lagrangian

$$\mathcal{L} = -\frac{1}{4} F_{\mu\nu} F^{\mu\nu} + (D^\mu \phi)^* D_\mu \phi - m^2 \phi^* \phi$$

where ϕ is a complex scalar field describing charged mesons and

$$D_\mu \phi = \partial_\mu \phi - i|e| A^a_\mu \phi$$

is the covariant derivative.

a) Calculate the generating functional

$$Z(\bar{J}, J) = \int D\phi^* D\phi e^{i \int d^4x \left[\left(\partial_\mu \phi^* \partial^\mu \phi - m^2 \phi^* \phi \right) + \bar{J}(x)\phi(x) + \phi^*(x) J(x) \right]}$$

by the appropriate shift. Treat ϕ and ϕ^* as independent variables (like ψ and $\bar{\psi}$, which are independent Grassman variables). Expanding the generating functional in powers of J and $\bar{J}$, find the propagators $\phi(x)\phi(y)$, $\phi^*(x)\phi^*(y)$, and $\phi(x)\phi^*(y)$ (some of them will be 0).

NB: to simlify your treatment, you may disregard the linear terms coming from the boundaries of the integration over time. (It will give you the expression for the propagator $(m^2 - p^2)^{-1}$ with unspecified singularity. A careful treatment of these boundary terms fixes the singularity in the form $(m^2 - p^2 - i\epsilon)^{-2}$, see the QM example on pp. 50-52 and the real KG propagator on p. 59)

b) Find vertices in this theory (for the set of Feynman rules for the reduced Green functions $\mathcal{G}$).

Problem 3

a)(8 points) Find the differential cross section of the $\pi\pi \to e^+e^-$ scattering in the Yukawa model in the lowest order in perturbation theory (the Lagrangian of the Yukawa model is $\mathcal{L} = \frac{1}{2}\partial_\mu\phi\partial^\mu\phi - \frac{m_\pi^2}{2}\phi^2 + \bar{\psi}(i\not\partial - m)\psi - g\phi\bar{\psi}\psi$, see p. 84)

b) Extra credit - 4 points.
Find the total cross section. (To simplify you calculation, you may assume that $s \gg m^2 = m_\pi^2$).

Problem 4 (12 points)

Pretending that there is no confinement, calculate the total cross section of $e^+e^- \to q\bar{q}$ annihilation in the lowest order in perturbation theory. Assume that there are only three flavors of quarks: u with charge $e_u = \frac{2}{3}$, d with $e_d = -\frac{1}{3}$, s with $e_s = -\frac{1}{3}$, and neglect their masses. The Lagrangian describing the interaction of quarks with photons is

$$\mathcal{L}_{\text{int}} = \sum_q e_q \bar{\psi}_q^k(x)\ \not A(x)\psi_q(x)$$

where A is the electromagnetic field.

Problem 5 (extra credit - 10 points)

A straight-line ordered gauge link in QCD is defined by the formula

$$[x, y] = \text{Pexp}\left\{ig\int_0^1 du\ (x-y)^\mu A_\mu(ux - uy + y)\right\}$$

where

$$\text{Pexp}\left\{\int_b^a du\ F(u)\right\} \stackrel{\text{def}}{\equiv} 1 + \int_b^a du\ F(u) + \int_b^a du\ F(u)\int_b^u du'\ F(u') + ...$$

(similarly to the Texp).

Prove that $\bar{\psi}(x)[x, y]\psi(y)$ is gauge invariant.

(Suggestion: (1) find the differential equation for $[\lambda n + x, x]$ with respect to λ (n is an arbitrary vector) and (2) using this differential equation prove that $[x, y] \to S(x)[x, y]S^\dagger(y)$ under the gauge rotation $A_\mu(x) \to S(x)A_\mu(x)S^\dagger(x) + \frac{i}{g}S(x)\partial_\mu(x)S^\dagger(x)$).