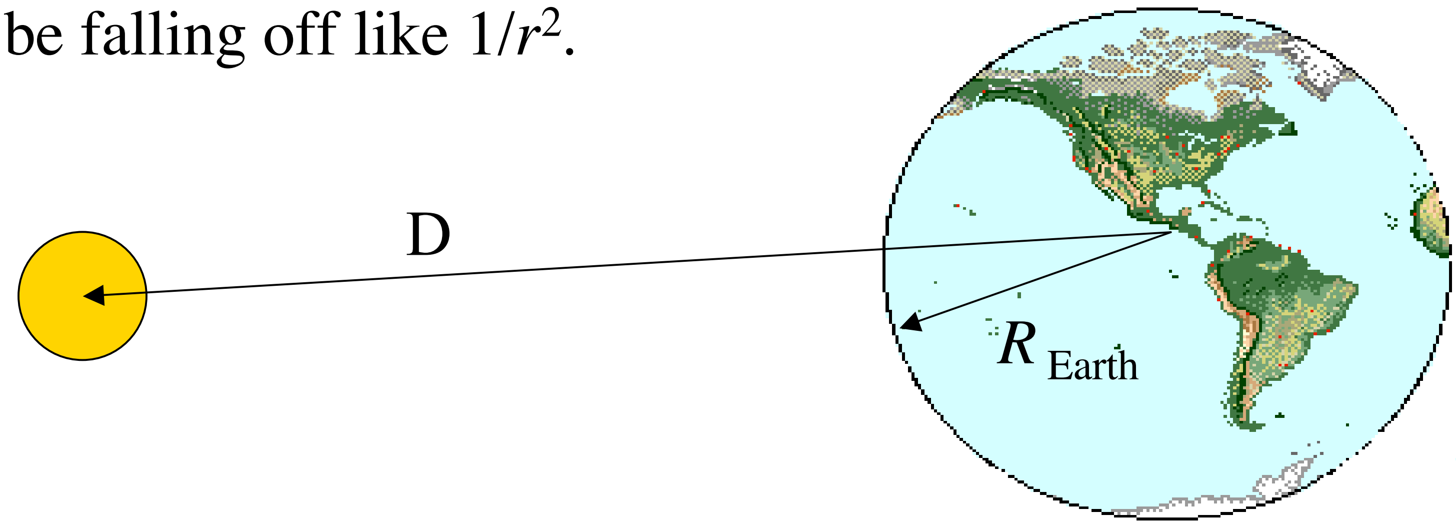
Gravitation

- Objects on the surface of Earth fall down with acceleration $a_{rad} = g = 9.81\ \text{m/s}^2$; Earth's radius is $R_{Earth} = 6380$ km.
- Moon circles the Earth once every 27.3 d = $2.36 \cdot 10^6$ s => $\omega = 2.66 \cdot 10^{-6}$/s . Moon is $D = 384{,}000$ km away => $a_{rad} = D\omega^2 = 0.00272\ \text{m/s}^2$ (3600 times smaller). D is 60 times bigger than R_{Earth} ! => Gravitational force must be falling off like $1/r^2$.



Newton's Law of Gravitation

- All masses are accelerated with the same acceleration at the same distance from Earth => $F \propto m/r^2$ (since $a = F/m$)
- Newton's 3rd law => Earth is attracted to any mass with force proportional to M => $F \propto m M/r^2$
- Need proportionality constant: G =>
  $F = G m M/r^2$
- Plug in numbers: on the surface of earth,
  $g = F/m = G M/R_{\text{Earth}}^2$; $M = 5.97 \cdot 10^{24}$ kg
  => $G = 6.7 \cdot 10^{-11}$ N m^2/kg^2
- Universal constant => Universal force law for any two bodies with masses m, M at distance r !

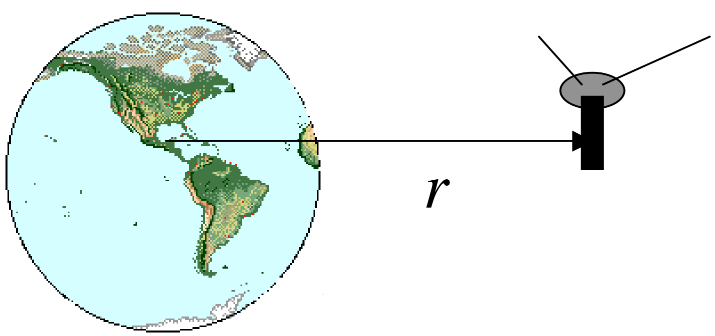
Important points

$$\vec{F}(\text{on } m_2 \text{ at } \vec{r}_2 \text{ due to } m_1 \text{ at } \vec{r}_1) = -G\frac{m_1 m_2}{|\vec{r}_2 - \vec{r}_1|^2}\frac{\vec{r}_2 - \vec{r}_1}{|\vec{r}_2 - \vec{r}_1|}$$

- Universal $1/r^2$ force law - describes not only gravity, but also electromagnetism …
- Valid not only for point masses, but for spherical extended masses as well
- G can be measured with torsion balance (but it's hard because it is so small) -> Value: $6.673 \cdot 10^{-11}$N m^2/kg^2
- Force always acts **on** center of gravity of an object (roughly equal to center of mass)
- Force always points **to** center of gravity of attracting mass (along distance vector)
- Mass *m* exerts equal and opposite force on mass *M* as that exerted by *M* on *m* .
- Superposition: Gravitational forces add

Example I

- Project "Lagrange Point"
  Put a satellite between Sun and Earth where the net gravitational force is zero
  - Satellite can be stationary (looking at Earth)
- Where to put?
  - $M_{\text{Earth}} = 6 \cdot 10^{24}$ kg , $M_{\text{Sun}} = 2 \cdot 10^{30}$ kg , $D_{\text{E-S}} = 1.5 \cdot 10^{11}$ m .
  - Require $GmM_{\text{Earth}}/r^2 = GmM_{\text{Sun}}/(D_{\text{E-S}} - r)^2$
    $\Rightarrow (D_{\text{E-S}} - r) = 577\ r \Rightarrow r = D_{\text{E-S}}/578 = 259{,}000$ km.



Example II

- Two steel balls floating in space (initially at rest). Masses $M = 10$ kg, $m = 5$ kg, $d = 0.1$ m apart.
  - Initial gravitational attraction: $F = G\, m\, M / d^2 = 3.34 \cdot 10^{-7}$ N (on each)
  - Initial acceleration: $3.34 \cdot 10^{-8}$ m/s^2 for M, $6.68 \cdot 10^{-8}$ m/s^2 for m.
  - After 60 s, first one has moved 0.06 mm. Second has moved 0.12 mm.
  - Center of mass remains at same point (1/3 of the way from first to second mass).

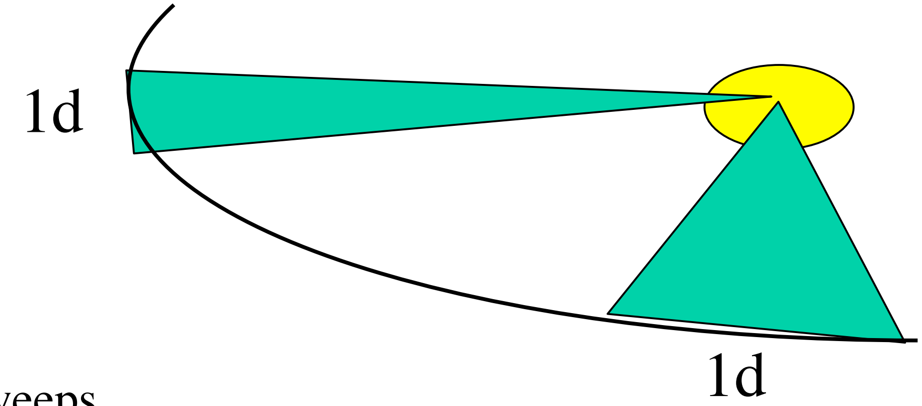
Gravitational Potential Energy

- Work done on a mass m in gravitational field of mass M (held fixed for now):
  $dW = -GmM/r^2\, dr$
- Note: only component dr of displacement contributes (component parallel to force)
- Integrate for finite distance:
  $\Delta W = -\int GmM/r^2\, dr = GmM\,(1/R_f - 1/R_i)$
  Note: it takes work to move from $R_i < R_f$
- Conservative Force! Can define potential energy:
  $\Delta U = -\Delta W = GmM\,(1/R_i - 1/R_f)$
- Choose reference point "infinitely far away" ($R_i = \infty$) =>
  $U_{grav} = -GmM/R_f$
- If both masses move, $U_{grav} = -GmM/D$
- Note: always negative!

Examples

- Two steel balls from before:
  (M = 10 kg, m = 5 kg, d = 0.1 m apart)
  - Initially $U_{grav} = -GmM/D = -3.34 \cdot 10^{-8}$ J
  - Touch at 1cm distance: $U_{grav} = -3.34 \cdot 10^{-7}$ J
  - Total kinetic energy increases by $3.0 \cdot 10^{-7}$ J
  - Mass m will have twice the speed as mass M (momentum conservation)
    => K.E. = $m/2\ v^2 + M/2\ v^2/4 = 3/4\ mv^2$
    => v = $8.9 \cdot 10^{-5}$ m/s (< 0.1 mm/s) (**not** a constant acceleration!)
- Escape velocity from Earth:
  - First $U_{grav} = -GmM_{Earth}/R_{Earth}$, K.E. = $m/2\ v^2$
  - Finally, $U_{grav} = 0$ and K.E. = 0
  - Energy conserved
    => $GmM_{Earth}/R_{Earth} = m/2\ v^2$
    => $v^2 = 2GM_{Earth}/R_{Earth}$ => v = 11,200 m/s
- Small displacement Δh => $\Delta U_{grav} = mg\ \Delta h$

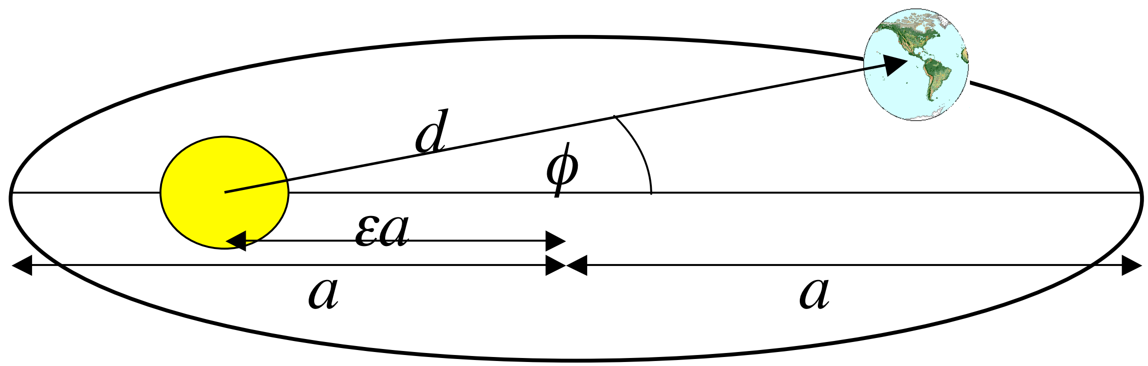
Motion of planets and Satellites



- Kepler's Laws:
  - Orbits of satellites (moons, planets,…) are elliptical
  - A line from the gravitating body (sun, planet,…) to the satellite sweeps out equal areas in equal times. (Conservation of angular momentum)
  - The period T of an orbit is proportional to $a^{3/2}$ (a = major half axis of ellipse) and $1/\sqrt{M}$.
- Circular orbit at distance R from central object with constant angular velocity ω
  - Centripetal force: $F_{\text{rad}} = m R \omega^2$
  - Gravitational force: $F_{\text{grav}} = GmM / R^2$
  - $\Rightarrow \omega^2 = 4\pi^2/T^2 = G M / R^3$
    $\Rightarrow T = 2\pi (R^3 / GM)^{1/2}$
  - Example: Satellite vs. Moon - Example: Miniature solar system

Motion of planets and Satellites (cont'd)

- Elliptical orbit



  - Gravitating body (sun, planet,…) located in one focal point of ellipse
  - Total " length" of ellipse is 2 *a* ("major half axis")
  - "Eccentricity" ε = distance from focal point to center/*a*
  - Major half axis *a* replaces R in 3rd Law:
    $T \propto a^{3/2}$
  - Conservation of angular momentum L requires $\omega \propto 1/d^2$ => "areas" law
  - => Kepler's Laws derived from Newton's Laws