

Motion in a Circle

- **Note:**
We will study the motion of a single object (mass point) on a circle around the origin. We will use a new set of variables to describe this motion:
 $\theta, \omega, \alpha, I, \mathbf{L}, \boldsymbol{\tau}$
and express equations of motion and K.E. in terms of these quantities.
- **Afterwards**, we will apply these ideas to rigid bodies rotating around a fixed axis.
- We will **deviate** from the book, where all these quantities are introduced in Chapters 9 and 10 using **mostly** rigid bodies right away.

“New” kinematic variables

- Particle going around the origin on a circle of radius R (not necessarily at constant speed) - assume x-y plane
- So far: described by $\mathbf{r}(t) = (x(t), y(t))$, $\mathbf{v}(t) = d\mathbf{r}/dt$,
 $\mathbf{a}(t) = d\mathbf{v}/dt$ (two components: a_{rad} , a_{tan})
- Now: use angle $\theta(t)$ [radians] to describe position:
 $(x, y) = (R \cos\theta, R \sin\theta)$
- Now: use angular velocity ω [radians/s] to describe velocity:
 $\omega = d\theta/dt$
Note: linear speed $|\mathbf{v}| = R \omega$
- Now: use angular acceleration α [radians/s²] to describe **tangential** acceleration: $\alpha = d\omega/dt$, $a_{\text{tan}} = R \alpha$
Note: $a_{\text{rad}} = R \omega^2$

Why...

- ...do we introduce new variables?
 - Simplify description: need only **one** number for position and **one** more for velocity (compared with usual position, velocity = vectors!)
 - Can apply what we learn to rotation of extended objects (spinning wheels, cylinders, fans, blades, tops,...)
 - Will discover new conservation law (important for astronomy, ice skaters, many other rotating objects, fundamental laws of Physics): Conservation of angular momentum **L**
 - Study new conditions for equilibrium (net torque = 0).

New dynamic variables

- Momentum? **Not** conserved ($a_{\text{rad}} \neq 0 \Rightarrow \mathbf{F}_{\text{net}} \neq 0 \Rightarrow$ external forces present).
- Kinetic Energy? **Yes** conserved **if** speed remains constant \Rightarrow
K.E. = $m/2 v^2 = m/2 (R\omega)^2 =$
 $1/2 (mR^2) \omega^2 = 1/2 I\omega^2$
- New quantity $I = mR^2 =$ moment of inertia
- Plays the same role as mass for linear motion
- Can be used to define an analog for momentum:
 $L = I\omega = mR^2\omega = mRv =$ angular momentum
- Proper definition as vector: $\mathbf{L} = m \mathbf{r} \times \mathbf{v}$ *)

*) Can be used to define angular velocity ω as a vector also: $\omega = \mathbf{L}/I$. That's why we call it angular **velocity** (not speed).

Examples

- Car moving around circle with constant speed v
 \Rightarrow const. $\omega = v/R$, $\alpha = 0$; \mathbf{L} and K.E. conserved
 - $v = 2\pi R / T \Rightarrow \omega = 2\pi / T$
 - $\theta(t) = \theta(0) + \omega t = \theta_0 + 2\pi t / T$
 - $x(t) = R\cos(\theta(t)) = R\cos(\theta_0 + \omega t)$; $y(t) = R\sin(\theta(t)) = R\sin(\theta_0 + \omega t)$.
 - $\mathbf{v}(t) = (-\omega R \sin(\theta_0 + \omega t), \omega R \cos(\theta_0 + \omega t))$; speed $|\mathbf{v}(t)| = \omega R$
 $\mathbf{a}(t) = (-\omega^2 R \cos(\theta_0 + \omega t), -\omega^2 R \sin(\theta_0 + \omega t)) = -\omega^2 (x, y)$; $|\mathbf{a}| = a_{rad} = \omega^2 R$
 - $I = mR^2$; $L = I \omega = mR^2 \omega = mRv$
 - $K.E. = 1/2 I \omega^2 = 1/2 mR^2 \omega^2 = 1/2 mv^2$
- Car moving around circle with constant angular acceleration α
 - $\omega = \omega(0) + \alpha t$ (\mathbf{L} and K.E. **not** conserved)
 - $\theta(t) = \theta(0) + \omega(0) t + 1/2 \alpha t^2$
- If you double the radius ($I \times 4$) and...
 - Leave ω const. $\Rightarrow v \times 2$, $L \times 4$, K.E. $\times 4$
 - Leave v const. $\Rightarrow \omega / 2$, $L \times 2$, K.E. =
 - Halve $v \Rightarrow \omega / 4$, $L =$, K.E. $/4$ Ball at end of string

Dynamics I

- Force can have tangential (F_{\parallel}) and radial (F_{rad}) components. Need $F_{\text{rad}} = mR\omega^2$ to keep R constant.
- Tangential force will change speed:
 - $m dv/dt = F_{\parallel}$ (Newton's 2nd Law)
 $= mR d\omega/dt = mR\alpha = I/R \alpha$
 - Introduce new quantity: Torque
 $\tau = R \cdot F_{\text{tan}} = I \alpha = mR^2 d\omega/dt = dL/dt$
 - Proper definition as a vector:
 $\boldsymbol{\tau} = \mathbf{r} \times \mathbf{F}$
 - Work done by torque:
 $\Delta \text{K.E.} = \Delta W = F_{\parallel} \Delta s = F_{\parallel} r \Delta\theta = \tau \Delta\theta$
 - Angular momentum change through torque:
 $\Delta v = a_{\parallel} \Delta t = F_{\parallel}/m \Delta t \Rightarrow \Delta L = mR \Delta v = RF_{\parallel} \Delta t = \tau \Delta t$

Dynamics II

- Now: F_{\parallel} zero, allow R to change
 - Torque is zero.
 - Assume R changes very slowly, so we still have $F_{\text{rad}} = mR\omega^2 \Rightarrow$
work done is $\Delta W = F_{\text{rad}} (-\Delta R) = -mR\omega^2 \Delta R = -m/2 \omega^2 \Delta R^2$
 - If R decreases, speed must increase! \rightarrow
 $\Delta W = \Delta \text{K.E.} = \Delta m/2 v^2 = mv \Delta v \Rightarrow$
 $mR\omega^2 (-\Delta R) = mv^2 (-\Delta R)/R = mv \Delta v \Rightarrow$
 $\Delta v/v = (-\Delta R)/R$; L is conserved!
 $(\Delta L = \Delta(mRv) = mv \Delta R + mR \Delta v)$
 - Proof in vector notation: $d\mathbf{L}/dt =$
 $d(m \mathbf{r} \times \mathbf{v})/dt = m d\mathbf{r}/dt \times \mathbf{v} + m \mathbf{r} \times d\mathbf{v}/dt =$
 $m \mathbf{v} \times \mathbf{v} + m \mathbf{r} \times \mathbf{a} = 0 + \mathbf{r} \times \mathbf{F} = \boldsymbol{\tau}$
 - K.E. is **not** conserved, but angular momentum **is** conserved (as long as no torque is present).

Comparison linear motion with angular motion

- Position: $x(t)$
- Velocity: v_x
- Acceleration: a_x
- Mass: m
- Force: F_x
- Linear momentum: $p_x = mv_x$
- Newton's Law:
 $F_x = ma_x = dp_x / dt$
- K.E. = $m/2 v^2 = 1/2m p^2$
- Momentum conserved if $\Sigma F_x = 0$
(and **no** "sideways force")
- Change of K.E.:
 $\Delta \text{K.E.} = \Delta W = F_x \Delta x$
- Position: $\theta(t)$
- Angular velocity: ω
- Ang. Acceleration: α
- Moment of Inertia: $I = mR^2$
- Torque: $\tau = R F_{\text{tan}}$
- Angular Momentum: $L = I \omega$
- "Newton's" Law:
 $\tau = I\alpha = dL / dt$
- K.E. = $I/2 \omega^2 = 1/2I L^2$
- L conserved always if $\Sigma \tau = 0$
(no tangential force)
- Change of K.E.:
 $\Delta W = \tau \Delta \theta (+ F_{\text{rad}} \Delta R)$

Summary: Motion is in 2D,
but can be described by
single (1D) variables